The third midterm covers Sections 14.7 and 15.1–15.7 of the textbook, except for the topics related to probability (pages 985–987).

The practice exam problems below are pretty representative of what you can expect for the midterm in terms of the difficulty level, nature of problems, and length of the exam, **but not in terms of specific questions or topics covered**. The latter is due to the simple statistical fact that the topics that can be covered on any given exam represent only a small random snapshot of the entire exam material, and another such random snapshot is likely to result in different questions and topics. Thus, simply studying the problems below will not adequately prepare you. The only way to be fully prepared for the exam is to work through the entire exam material, including all assigned homework.

**Practice Exam Problems:**

1. Find all critical points of the function

   \[ f(x, y) = x^2 + y^2 + x^2y \]

   and classify them as local maxima, local minima, or saddle points.

**Solution:** We first find the critical points. The gradient of \( f \) is

\[
\nabla f(x, y) = (2x + 2xy, 2y + x^2).
\]

Setting this equal to the zero vector gives the equations for the critical points:

\[
2x + 2xy = 0; \quad 2y + x^2 = 0. \tag{1}
\]

To solve this system, we first use (2) to get \( y = -\frac{x^2}{2} \). Next substitute this into (1) to get \( 2x - x^3 = 0 \), or \( x^3 = 2x \), i.e., \( x(x^2 - 2) = 0 \). The latter equation has three solutions:

\[
x = 0, \quad x = \sqrt{2}, \quad x = -\sqrt{2}. \tag{2}
\]

From (1), we get the \( y \)-values corresponding to these solutions:

\[
y = 0, \quad y = -1, \quad y = -1.
\]

Hence we have three critical points: \( (0, 0), (\sqrt{2}, -1), (-\sqrt{2}, -1) \)
Next, we apply the 2nd derivative test to each of these points in order to determine whether it is a saddle point, a local maximum or a local minimum. We compute:

\[ f_{xx} = 2 + 2y, \quad f_{yy} = 2, \quad f_{xy} = f_{yx} = 2x, \]

\[ f_{xx} f_{yy} - f_{xy}^2 = 4 + 4y - 4x^2. \]

The condition for a saddle point is that \( f_{xx} f_{yy} - f_{xy}^2 < 0 \), for a local maximum/minimum it is \( f_{xx} f_{yy} - f_{xy}^2 > 0 \), with a maximum occurring if \( f_{xx} < 0 \) and a minimum if \( f_{xx} > 0 \). Substituting the three critical points into the above formulae, we get:

- At \((0, 0)\): \( f_{xx} = 2 > 0 \), \( f_{xx} f_{yy} - f_{xy}^2 = 4 > 0 \), so \((0, 0)\) is a local minimum.
- At \((\sqrt{2}, -1)\): \( f_{xx} f_{yy} - f_{xy}^2 = -8 < 0 \), so \((\sqrt{2}, -1)\) is a saddle point.
- At \((-\sqrt{2}, -1)\): \( f_{xx} f_{yy} - f_{xy}^2 = -8 < 0 \), so \((-\sqrt{2}, -1)\) is a saddle point.

**Remarks:** When solving for the critical points, one has to take into account the possibility that the variables are 0. In particular, one cannot simply cancel a variable from an equation without first analyzing the case when this variable is equal to zero. Sometimes, this has no consequences on the end result, but in the above problem it makes a crucial difference: if one divides (1) by \( x \), one misses out on the case \( x = 0 \), and the point \((0, 0)\), which is a local minimum.

Also note that the equation \( x^2 = 2 \) has two solutions, \( x = \sqrt{2} \) and \( x = -\sqrt{2} \), yielding two points, \((\sqrt{2}, -1)\) and \((-\sqrt{2}, -1)\).

2. Evaluate the integral \( \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy \). (Hint: Sketch the region and reverse the order of integration.)

**Solution:** First read off the integration limits in the given integral: \( 0 \leq y \leq 1, \sqrt{y} \leq x \leq 1 \). Next, sketch the region determined by these inequalities; it is the region below the parabola \( y = x^2 \) and above the \( x \)-axis, between \( x = 0 \) and \( x = 1 \). From the picture of this region, we can
read off the inequalities needed for reversing the order of integration, i.e., with constant limits on \( x \):

\[ 0 \leq x \leq 1, \quad 0 \leq y \leq x^2. \]

Thus, the given integral is

\[
\int_0^1 \int_0^{x^2} \sqrt{x^3 + 1} \, dy \, dx = \int_0^1 x^2 \sqrt{x^3 + 1} \, dx.
\]

Substituting

\[ u = x^3 + 1, \quad du = 3x^2 \, dx \]

gives for this integral

\[
\int_1^2 \frac{1}{3} \sqrt{u} \, du = \left. \frac{2}{9} u^{3/2} \right|_1^2 = \frac{2}{9} (2\sqrt{2} - 1).
\]

**Remark:** Note that, in the above form, the integral cannot be computed, since the function \( \sqrt{x^3 + 1} \) has no (elementary) anti-derivative. Thus, the only way to evaluate the integral is by reversing the order of integration.

3. Evaluate the integral \( \int \int_R xy \, dA \), where \( R \) is the region in the first quadrant that lies between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \) (i.e., a quarter annulus).

**Solution:** In polar coordinates, \( R \) is given by the inequalities

\[ 0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 2, \]

and \( dA = r \, dr \, d\theta \). Thus,

\[
\int \int_R xy \, dA = \int_0^{\pi/2} \int_1^2 (r \cos \theta)(r \sin \theta) r \, dr \, d\theta
\]

\[ = \int_0^{\pi/2} \left( \cos \theta \sin \theta \left( \frac{1}{4} r^4 \right) \right) \bigg|_1^2 \, d\theta
\]

\[ = \frac{2^4 - 1^4}{4} \int_0^{\pi/2} u \, du \quad \text{(substitute} \ u = \sin \theta, \ du = \cos \theta d\theta)\]

\[ = \frac{15}{4} \left( \frac{1}{2} u^2 \right) \bigg|_0^1 = \frac{15}{8}.
\]

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Remarks: Note that the region \( R \) is not the same as the region given by the inequalities \( 0 \leq x \leq 2, \sqrt{1 - x^2} \leq y \leq \sqrt{4 - x^2} \), since the lower bound for \( y, \sqrt{1 - x^2} \), is only valid when \( 0 \leq x \leq 1 \), while for \( 1 < x \leq 2 \), the lower bound is 0, not \( \sqrt{1 - x^2} \) (which is not defined in this range). (To see this, draw a picture of the region \( R \), then read off the limits for \( y \) as the lower and upper boundary curves for \( R \).)

To do this integral in rectangular coordinates, one would therefore have to split the \( x \)-range into two parts \( 0 \leq x \leq 1 \) and \( 1 \leq x \leq 2 \) and treat them separately. This, of course, leads to messy computations, so using polar coordinates as shown above is the appropriate method here.

4. Find the volume of the region that lies inside the cylinder \( x^2 + y^2 = 1 \), below the plane \( z = 4 \), and above the surface \( z = 1 - x^2 - y^2 \).

Solution: It is easiest to compute the volume via double integrals. The given region is the region lying over the unit disk \( D : x^2 + y^2 \leq 1 \) in the \( xy \)-plane, and between the functions

\[
f(x, y) = 4, \quad g(x, y) = 1 - x^2 - y^2.
\]

Its volume is therefore given by the formula

\[
V = \int \int_D (f(x, y) - g(x, y)) dA = \int \int_D (3 + x^2 + y^2) dA
\]

\[
= \int_0^{2\pi} \int_0^1 (3 + r^2)r \, dr \, d\theta = \int_0^{2\pi} \left( \frac{3}{2}r^2 + \frac{1}{4}r^4 \right) \bigg|_0^1 \, d\theta
\]

\[
= \int_0^{2\pi} \frac{7}{4} \, d\theta = \frac{7}{4} \left. \right|_0^{2\pi} = \frac{7\pi}{2}
\]

(Alternatively, one could compute the volume as a triple integral

\[
V = \int \int \int 1 \, dV.
\]

Using cylindrical coordinates, the integral becomes

\[
\int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r \left|_1^{1-r^2} \right. dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 r(3 + r^2) \, dz \, dr \, d\theta,
\]
which is the same as the above double integral in $r$ and $\theta$.)

5. (a) Express the point $(x, y, z) = (1, 1, 1)$ in cylindrical coordinates.

Solution: 

$$ (r, \theta, z) = \left( \sqrt{x^2 + y^2}, \arctan \frac{y}{x}, z \right) = \left( \sqrt{2}, \frac{\pi}{4}, 1 \right) $$

(b) Convert the equation (in cylindrical coordinates) 

$$ r = 2 \cos \theta $$

to rectangular coordinates and identify the surface represented by the equation. (Be specific, saying, for example, “a half cone, centered at the origin, whose surface forms an angle $\frac{\pi}{6}$ with the positive $z$-axis”, instead of “a cone”.

Solution: Multiplying both sides of the given equation by $r$, we get $r^2 = 2r \cos \theta$, or $x^2 + y^2 = 2x$. Completing the square in $x$, we get $(x - 1)^2 + y^2 = 1$, which represents a vertical cylinder of radius 1, whose axis is the line given by the parametric equations $x = 1, y = 0, z = t$.

6. The following problems are independent of one another. In each case, set up, but do not evaluate, an iterated integral in the coordinate system specified. (Hint: You may want to sketch the appropriate regions to determine the correct integration limits.)

(a) Express, as a double integral in rectangular coordinates, the volume of a tetrahedron with corners $(0, 0, 0), (1, 0, 0), (0, 2, 0),$ and $(0, 0, 3)$.

Solution: Sketching the tetrahedron, we see that the coordinate planes in the first octant form three of its faces, while the fourth face, the “roof”, is given by the plane through the points $(1, 0, 0), (0, 2, 0), (0, 0, 3)$. This is the plane with $x$, $y$, $z$-intercepts 1, 2, 3, so its equation is $x + \frac{y}{2} + \frac{z}{3} = 1$ or $z = 3 - 3x - \frac{3y}{2}$. The “floor” of the tetrahedron lies in the $xy$-plane, i.e., has $z$-coordinate $z = 0$, and is the triangle determined by the points $(0, 0), (1, 0) and (0, 2)$ in the $xy$-plane. A sketch of this triangle shows it is described by the inequalities $0 \leq x \leq 1, 0 \leq y \leq 2 - 2x$. 

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Thus the double integral representing the volume of the tetrahedron is

\[ V = \int_0^1 \int_0^{2-2x} \left( 3 - 3x - \frac{3y}{2} \right) \, dy \, dx \]

(b) Express, as a **double integral in polar coordinates**, the volume of the “spherical cap” consisting of the portion of the solid sphere of radius 2 centered at the origin that lies above the plane \( z = 1 \).

**Solution:** The plane \( z = 1 \) and the sphere \( x^2 + y^2 + z^2 = 4 \) intersect when \( r^2 = x^2 + y^2 = 3 \), or \( r = \sqrt{3} \). Thus, the region over which the spherical cap lies is a disk of radius \( \sqrt{3} \), centered at the origin (not radius 2, since that would be the disk over which the entire sphere lies). Moreover, the “top” and “bottom” surfaces of this spherical cap are given by \( z = \sqrt{4 - r^2} \) and \( z = 1 \) respectively.

Thus we get

\[ V = \int_0^{2\pi} \int_0^{\sqrt{3}} (\sqrt{4 - r^2} - 1) \, r \, dr \, d\theta \]

(c) Express, as a **double integral**, the mass \( m \) of a lamina occupying the triangle with vertices \((0, 0)\), \((0, 3)\), and \((1, 1)\), if the density at any point inside the triangle is equal to the distance of that point from the \( x \)-axis.

**Solution:** The distance of a point \((x, y)\) to the \( x \)-axis is \( y \), so the density function is \( \rho(x, y) = y \). The given triangle is described by the inequalities \( 0 \leq x \leq 1 \), \( x \leq y \leq 3 - 2x \). (Draw picture!) Hence the mass of the lamina is

\[ m = \int \int_D \rho(x, y) \, dA = \int_0^1 \int_x^{3-2x} y \, dy \, dx \]